

Formulation of Leggett–Garg inequalities in terms of q -entropies

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As is well known, the macroscopic realism and the noninvasive measurability together lead to Leggett–Garg inequalities violated by quantum mechanics. We consider tests of the Leggett–Garg type with use of the q -entropies. For all $q \geq 1$, quantum mechanics predicts violations of an entire family of q -entropic inequalities of the Leggett–Garg type. Violations are exemplified with a quantum spin- s system. In general, entropic Leggett–Garg inequalities give only necessary conditions that some probabilistic model is compatible with the macrorealism in the broader sense. The presented q -entropic inequalities allow to widen a class of situations, in which an incompatibility with the macrorealism can be tested. We also examine q -entropic inequalities of the Leggett–Garg type in the case of detection inefficiencies.

Keywords: Bell theorem, Leggett–Garg inequalities, macrorealism, noninvasive measurability, conditional q -entropies

I. INTRODUCTION

Physicists know a few key advances that emphasize distinctions of the quantum world from the classical one. The uncertainty principle was a primary among them [1]. The Bell theorem [2] is a next profound insight into the subject. It is closely related to the Einstein–Podolsky–Rosen question [3] and later reformulation by Bohm [4]. Studies of foundations of quantum theory are now connected with a progress in quantum information processing [5]. Violations of Bell inequalities reveal non-classical features of correlations between spatially-separated quantum systems [6]. The Clauser–Horne–Shimony–Holt (CHSH) scenario [7] is the first setup tested in experiments [8, 9]. Violations of the CHSH inequality imply that predictions of quantum theory are not compatible with the local realism [10]. The Klyachko–Can–Binicioğlu–Shumovsky (KCBS) scenario [11] pertain to the measurement statistics of a single spin-1 system. Results of existing experiments stimulated a certain interest to non-local hidden-variable theories [12].

Leggett–Garg inequalities [13] form one of directions inspired by the Bell theorem. these inequalities are based on the following two concepts often called the macrorealism in the broader sense. First, we assume that physical properties of a macroscopic object preexist irrespectively to the act of observation. Second, measurements are noninvasive in the sense that the measurement of an observable at any instant of time does not alert its subsequent evolution. Consequences of the assumptions were originally examined by Leggett and Garg [13]. They are commonly known as Leggett–Garg inequalities [14]. It turns out that predictions of quantum mechanics lead to violations of these inequalities. Leggett–Garg inequalities are now the subject of active experimental [15–17] and theoretical investigations [18–22]. Interesting physical proposals are discussed in [23–25].

Entropic approach to formulating the Bell theorem was proposed in [26] and later studied in [27–29]. In particular, entropic inequalities of the Bell type were derived for the KCBS scenario [28, 29]. Entropic formulations are very useful due to the following. First, they can deal with any finite number of outcomes. Second, entropic approach allows to analyze more realistic cases with detection inefficiencies [28]. Additional possibilities to analyze non-locality or contextuality or probabilistic models are provided by use of the q -entropies [30]. Using the q -entropic inequalities, we can widen a class of probability distributions, for which the non-locality or contextuality are testable. It is an alternative to the approach with adding some shared randomness [31]. Further, the q -entropic inequalities are expedient in analyzing cases with detection inefficiencies [30].

Leggett–Garg tests probe the correlations of a single system measured at different times. It is appealing to study restrictions of the Leggett–Garg type within an entropic approach. Using standard entropic functions, such an analysis has been carried out by the writers of [32]. In the present paper, we aim to study restrictions of the Leggett–Garg type with formulating them in terms of Tsallis q -entropies. Our paper is organized as follows. In Section II, we recall basic properties of the q -entropies. Leggett–Garg inequalities in terms of q -entropies are derived in Section III. We also consider a formulation of entropic Leggett–Garg inequalities in the case of detection inefficiencies. In Section IV, violations of the derived Leggett–Garg inequalities are exemplified with a quantum spin- s system. We also discuss trade-offs between violations of the q -entropic inequalities and the required efficiency of detectors. In Section V, we

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conclude the paper with a summary of results.

II. TSALLIS q -ENTROPIES AND THEIR PROPERTIES

In this section, we recall some preliminary material on the Tsallis q -entropies and their properties. Let X be discrete random variable taking values according to the probability distribution $\{p(x) : x \in \Omega_X\}$. The Tsallis entropy of degree $q > 0 \neq 1$ is defined by [33]

$$H_q(X) := \frac{1}{1-q} \left(\sum_{x \in \Omega_X} p(x)^q - 1 \right). \quad (1)$$

With slightly other factor, this entropic function was proposed by Havrda and Charvát [34]. Let Y be another variable taking values according to the probability distribution $\{p(y) : y \in \Omega_Y\}$. The joint q -entropy $H_q(X, Y)$ is defined similarly to (1), but with joint probabilities $p(x, y)$. We can rewrite the entropy (1) in the form

$$H_q(X) = - \sum_{x \in \Omega_X} p(x)^q \ln_q p(x) = \sum_{x \in \Omega_X} p(x) \ln_q \left(\frac{1}{p(x)} \right). \quad (2)$$

Here, the q -logarithm $\ln_q(\xi) = (\xi^{1-q} - 1)/(1 - q)$ is defined for $q > 0 \neq 1$ and $\xi > 0$. In the limit $q \rightarrow 1$, we obtain $\ln_q(\xi) \rightarrow \ln \xi$ and the Shannon entropy

$$H_1(X) = - \sum_{x \in \Omega_X} p(x) \ln p(x). \quad (3)$$

For brevity, we will usually omit the range of summation. The entropy (1) is widely used in many disciplines [35]. The Rényi entropies [36] form another especially important family of generalized entropies. Applications of these entropies and their quantum counterparts are considered in the book [37].

To consider cases with detector inefficiencies, the following question will rise [28]. In real experiments, we do not deal immediately with original distributions of the form $\{p(x)\}$. Such distributions will somehow be altered due to detector inefficiencies. To the given $\eta \in [0; 1]$ and probability distribution $\{p(x) : x \in \Omega_X\}$, we assign another probability distribution

$$\{\eta p(x) : x \in \Omega_X\} \cup \{1 - \eta\}. \quad (4)$$

This probability distribution corresponds to some “altered” random variable X_η . For all $q > 0$, the entropy $H_q(X_\eta)$ can be expressed as [30]

$$H_q(X_\eta) = \eta^q H_q(X) + h_q(\eta). \quad (5)$$

As usually, the binary q -entropy reads

$$h_q(\eta) := -\eta^q \ln_q \eta - (1 - \eta)^q \ln_q (1 - \eta). \quad (6)$$

From three probability distributions, we can built another probability distribution

$$\{p_{\eta\eta}\} := \{\eta^2 p(x)\} \cup \{\eta(1 - \eta)p(y)\} \cup \{\eta(1 - \eta)p(z)\} \cup \{(1 - \eta)^2\}. \quad (7)$$

It is assigned to some random variable X_η . For all $q > 0$, we then have [30]

$$H_q(X_{\eta\eta}) = \eta^{2q} H_q(X) + \eta^q (1 - \eta)^q (H_q(Y) + H_q(Z)) + (\eta^q + (1 - \eta)^q + 1) h_q(\eta). \quad (8)$$

We will use the results (5) and (8) for studying entropic Leggett–Garg inequalities in the case of detection inefficiencies.

Like the Braunstein–Caves inequality [26], entropic Leggett–Garg inequalities are formulated in terms of the conditional entropy [32]. The entropy of X conditional on knowing Y is defined as [40]

$$H_1(X|Y) := \sum_y p(y) H_1(X|y) = - \sum_x \sum_y p(x, y) \ln p(x|y). \quad (9)$$

Here, we take $H_1(X|y) := - \sum_x p(x|y) \ln p(x|y)$ and $p(x|y) = p(x, y) p(y)^{-1}$ according to the Bayes rule. The quantity (9) is the standard conditional entropy. For partitions on quantum logic, the standard conditional entropies were studied in [38]. Further development with use of the Rényi and Tsallis entropies was reported in [39].

We recall the q -entropic extension of (9). Introducing the particular functional

$$H_q(X|y) := - \sum_x p(x|y)^q \ln_q p(x|y) , \quad (10)$$

we define the conditional q -entropy as [41, 42]

$$H_q(X|Y) := \sum_y p(y)^q H_q(X|y) . \quad (11)$$

taking the limit $q \rightarrow 1$, this definition leads to (9). Below, we will mainly use the following properties of (11). For all $q > 0$, the entropy (11) satisfies

$$H_q(X, Y) = H_q(Y|X) + H_q(X) = H_q(X|Y) + H_q(Y) . \quad (12)$$

It is usually referred to as the chain rule for the conditional q -entropy [41]. Due to theorem 2.4 of [41], we have the chain rule with any finite number of random variables:

$$H_q(X_1, X_2, \dots, X_n) = \sum_{j=1}^n H_q(X_j|X_{j-1}, \dots, X_1) . \quad (13)$$

For real $q \geq 1$ and integer $n \geq 1$, the conditional q -entropy also satisfies [30, 41]

$$H_q(X|Y_1, \dots, Y_{n-1}, Y_n) \leq H_q(X|Y_1, \dots, Y_{n-1}) . \quad (14)$$

According to (14), conditioning on more can only reduce the q -entropy of degree $q \geq 1$. In the paper [30], we examined formulation of Bell's theorem in terms of q -entropies. In a similar manner, we will study the macrorealism in the broader sense with use of the conditional q -entropies. Deriving q -entropic forms of Leggett–Garg inequalities will be based on the properties listed above.

III. ENTROPIC LEGGETT–GARG INEQUALITIES IN TERMS OF q -ENTROPIES

We begin with discussion of basic points involved in the macrorealistic picture. Leggett–Garg inequalities are based on the following two assumptions known as the macroscopic realism and the noninvasive measurability at the macroscopic level [14]. We consider a macrorealistic system, in which $X(t_j)$ is a dynamical variable at the time moment t_j . Formally, the macroscopic realism *per se* implies that outcomes x_j of the variables $X(t_j)$ at all instants of time preexist irrespective of their measurements. The noninvasive measurability means that the act of measuring $X(t_j)$ at an earlier time t_j does not affect its subsequent value at a later time $t_k > t_j$. These assumptions lead to the following conclusion. For each particular choice of time instants, the statistics of outcomes is described by a joint probability distribution $p(x_1, x_2, \dots, x_n)$. The joint probabilities are expressed as a convex combination of the form [43, 44]

$$p(x_1, x_2, \dots, x_n) = \sum_{\lambda} \varrho(\lambda) P(x_1|\lambda) P(x_2|\lambda) \cdots P(x_n|\lambda) . \quad (15)$$

Here, the product of conditional probabilities $P(x_j|\lambda)$ is averaged over a hidden-variable probability distribution. Of course, in any macrorealistic model the probabilities $P(x_j|\lambda) \geq 0$ should obey

$$\sum_{x_j} P(x_j|\lambda) = 1 . \quad (16)$$

Further, unknown hidden-variable probabilities $\varrho(\lambda) \geq 0$ should satisfy $\sum_{\lambda} \varrho(\lambda) = 1$. By a structure, the n -variable distribution (15) will marginalize to particular distributions with lesser number of variables. This is a consistency condition for macrorealistic models.

Like probabilistic model of the local realism and noncontextuality, the existence of joint probability distributions of the form (15) does result in certain inequalities between conditional entropies. Entropic inequalities of the paper [32] were derived similarly to the treatment given by Braunstein and Caves [26]. For the CHSH and KCBS scenarios, the q -entropic inequalities were formulated in [30]. Let us apply these ideas to macrorealistic models. We will use X_j as shortening for $X(t_j)$. For brevity, we consider the case $n = 3$ with the variables X_1, X_2, X_3 .

$$\begin{aligned} H_q(X_1, X_3) &\leq H_q(X_1, X_2, X_3) = H_q(X_1) + H_q(X_2|X_1) + H_q(X_3|X_2, X_1) \\ &\leq H_q(X_1) + H_q(X_2|X_1) + H_q(X_3|X_2) . \end{aligned} \quad (17)$$

Here, we used the chain rule (13) and suitable relations of the form (14). Subtracting $H_q(X_1)$ and using the chain rule again, one gives the entropic inequality

$$H_q(X_3|X_1) \leq H_q(X_3|X_2) + H_q(X_2|X_1) , \quad (18)$$

which holds for $q \geq 1$. For $q = 1$, this formula is reduced to the Shannon-entropy inequality given in [32]. By a parallel argument, for real $q \geq 1$ and integer $n \geq 3$ we obtain

$$H_q(X_n|X_1) - \sum_{n \geq j \geq 2} H_q(X_j|X_{j-1}) =: \mathcal{C}_q \leq 0 . \quad (19)$$

We introduce here the characteristic quantity \mathcal{C}_q . In the next section, we will exemplify that quantum mechanics sometimes leads to violations of (19). Positive values of \mathcal{C}_q then characterize an amount with which entropic Leggett–Garg inequalities are violated. In the case $n = 4$, the relation (19) is formally similar to the q -entropic version of the Braunstein–Caves inequality. For $n = 5$, the result (19) mathematically coincides with the q -entropic inequalities holding for non-contextual models in the KCBS scenario. Such q -entropic inequalities for both the CHSH and KCBS scenarios were examined in [30].

Real measurement devices are inevitably exposed to noise. Entropic approach allows to take into account such a feature. The Shannon-entropy formulation of Bell’s theorem with detection inefficiencies was considered in [28]. In the paper [30], we extended this treatment to q -entropic inequalities. It is relevant to address the question of detection inefficiencies also for entropic Leggett–Garg inequalities. For these purposes, we adopt one of the inefficiency models considered in [28]. Let us assume that the no-click event can occur in each act of observation irrespectively to other observations. We also assume that detectors are all of efficiency $\eta \in [0; 1]$. For a pair of outcomes of the dynamical quantities X and Y , we have probabilities

$$p^{(\eta\eta)}(x, y) = \eta^2 p(x, y) , \quad (20)$$

$$p^{(\eta\eta)}(x, \emptyset) = \eta(1 - \eta)p(x) , \quad (21)$$

$$p^{(\eta\eta)}(\emptyset, y) = \eta(1 - \eta)p(y) , \quad (22)$$

$$p^{(\eta\eta)}(\emptyset, \emptyset) = (1 - \eta)^2 . \quad (23)$$

Here, the no-click event is denoted by “ \emptyset ”. The two-variable probability distribution (20)–(23) marginalizes to the single-observable distributions of the form

$$p^{(\eta)}(x) = \eta p(x) , \quad (24)$$

$$p^{(\eta)}(\emptyset) = 1 - \eta . \quad (25)$$

Let us rewrite (19) without conditional entropies. Using the chain rule (12), we finally obtain the theoretical result

$$0 \leq -\mathcal{C}_q = \sum_{j=1}^{n-1} H_q(X_j, X_{j+1}) - H_q(X_1, X_n) - \sum_{k=2}^{n-1} H_q(X_k) . \quad (26)$$

All the entropies involved in (26) pertain to the inefficiency-free case, when $\eta = 1$. However, we actually deal with “altered” probability distributions described by the formulas (20)–(23) and (24)–(25). Using the results (5) and (8), we obtain

$$H_q^{(\eta)}(X_k) = \eta^q H_q(X_k) + h_q(\eta) . \quad (27)$$

$$\begin{aligned} H_q^{(\eta\eta)}(X_j, X_{j+1}) &= \eta^{2q} H_q(X_j, X_{j+1}) + \eta^q (1 - \eta)^q (H_q(X_j) + H_q(X_{j+1})) \\ &\quad + (\eta^q + (1 - \eta)^q + 1) h_q(\eta) . \end{aligned} \quad (28)$$

By $H_q^{(\eta)}(X_k)$ and $H_q^{(\eta\eta)}(X_j, X_{j+1})$, we mean the actual q -entropies calculated with the probability distributions (24)–(25) and (20)–(23). Instead of the characteristic quantity \mathcal{C}_q , we will deal with

$$\mathcal{C}_q^{(\eta\eta)} := H_q^{(\eta\eta)}(X_n, X_1) - \sum_{j=1}^{n-1} H_q^{(\eta\eta)}(X_j, X_{j+1}) + \sum_{k=2}^{n-1} H_q^{(\eta)}(X_k) . \quad (29)$$

By calculations, we obtain the relations

$$\mathcal{C}_q^{(\eta\eta)} = \eta^{2q}\mathcal{C}_q - \Delta_q(\eta) , \quad (30)$$

$$\Delta_q(\eta) = \eta^q (\eta^q + 2(1-\eta)^q - 1) \sum_{k=2}^{n-1} H_q(X_k) + (n-2)(\eta^q + (1-\eta)^q) h_q(\eta) . \quad (31)$$

For the case $n = 5$, the additional term (31) was examined in [30]. For $q > 1$, the factor $\eta^q + 2(1-\eta)^q - 1$ is negative for some values of η near 1 from below. Thus, the first term in the right-hand side of Eq. (31) can take positive or negative values. The second term in the right-hand side of Eq. (31) is certainly positive.

The Leggett–Garg inequality (19) implies $\mathcal{C}_q \leq 0$. Using measurement statistics, we have to analyze the quantity (30). Assume that measurement data have lead to the result $\mathcal{C}_q^{(\eta\eta)} > 0$. In principle, we still cannot claim $\mathcal{C}_q > 0$. We must also confide that the violating term $\eta^{2q}\mathcal{C}_q$ is sufficiently larger than the additional term (31). For very high values of efficiency parameter η , the term (31) will be small. At the same time, values of $\Delta_q(\eta)$ also depend on the entropic parameter $q \geq 1$. It is instructive to consider these questions within concrete examples. We will thoroughly address the issue in the next section.

IV. ENTROPIC LEGGETT–GARG INEQUALITIES FOR SPIN SYSTEMS

In this section, we consider concrete quantum-mechanical examples, in which the q -entropic Leggett–Garg inequalities are violated. Following [32], we consider a quantum spin- s system. Initially, it is prepared in the completely mixed state

$$\rho_* = \frac{1}{2s+1} \sum_{m=-s}^{+s} |s, m\rangle \langle s, m| = \frac{1}{2s+1} \mathbb{1} . \quad (32)$$

As usually, the states $|s, m\rangle$ are common eigenstates of the commuting operators $\mathbf{S}^2 = \mathbf{S}_x^2 + \mathbf{S}_y^2 + \mathbf{S}_z^2$ and \mathbf{S}_z . By $\mathbb{1}$, we denote the identity operator in $(2s+1)$ -dimensional Hilbert space of the system. We use the standard notation

$$\mathbf{S}^2 |s, m\rangle = s(s+1)\hbar^2 |s, m\rangle , \quad \mathbf{S}_z |s, m\rangle = m\hbar |s, m\rangle . \quad (33)$$

Evolution of the system in time is generated by the Hamiltonian [32]

$$\mathbf{H} = \omega \mathbf{S}_y . \quad (34)$$

We will consider measurements of z -component of the spin. In the Heisenberg picture, its evolution is represented as

$$\mathbf{X}(t) = \mathbf{V}(t) \mathbf{S}_z \mathbf{V}(t)^\dagger , \quad \mathbf{V}(t) = \exp(-i t \hbar^{-1} \mathbf{H}) . \quad (35)$$

In this picture, the state (32) remains unchanged until the act of observation occurs. For brevity, we introduce rank-one projectors of the form

$$\Pi_m(t) := \mathbf{V}(t)^\dagger |s, m\rangle \langle s, m| \mathbf{V}(t) . \quad (36)$$

With the initial state (32), the observation of $\mathbf{X}(t)$ at the instant $t = t_j$ leads to the result $m \in \{-s, -s+1, \dots, +s\}$ with probability

$$p(m) = \text{Tr}(\rho_* \Pi_m(t_j)) = (2s+1)^{-1} . \quad (37)$$

Thus, the outcomes are all equiprobable. Due to the projection postulate, the normalized post-measurement state is written as

$$p(m)^{-1} \Pi_m(t_j) \rho_* \Pi_m(t_j) = \Pi_m(t_j) . \quad (38)$$

Hence, the context for next observation is determined. Calculating the conditional probability of obtaining the outcome m' at the next time t_k , we write

$$p(m'|m) = \text{Tr}(\Pi_m(t_j) \Pi_{m'}(t_k)) = |\langle s, m' | \exp(-i \theta_{kj} \hbar^{-1} \mathbf{S}_y) | s, m \rangle|^2 , \quad (39)$$

where $\theta_{kj} = \omega(t_k - t_j)$. Probabilities (39) are immediately expressed in terms of elements of the corresponding rotation matrix. These elements also known as the Wigner small d -functions are defined as [45]

$$d_{m',m}^{(s)}(\theta) = \langle s, m' | \exp(-i\theta \hbar^{-1} S_y) | s, m \rangle . \quad (40)$$

The elements of rotation matrices are well tabulated. Hence, we write a useful expression

$$p(m'|m) = |d_{m',m}^{(s)}(\theta)|^2 . \quad (41)$$

Combining (37) and (41) gives the joint probability distribution for outcomes of two sequential measurements:

$$p(m, m') = p(m) p(m'|m) = \frac{1}{2s+1} |d_{m',m}^{(s)}(\theta_{kj})|^2 . \quad (42)$$

Recall that expressions of such a kind have been obtained for the CHSH scenario with singlet initial state of the spin- s system [26].

Following [32], we now consider equidistant time intervals. That is, two sequential measurements of $\mathbf{X}(t)$ are separated by the interval $t_j - t_{j-1} = \Delta t$. For brevity, we introduce two parameters $\Delta\theta = \omega\Delta t$ and $\theta = (n-1)\Delta\theta$, and an auxiliary function

$$F_q^{(s)}(\theta) := \frac{(2s+1)^{-q}}{q-1} \sum_{m=-s}^{+s} \left(1 - \sum_{m'=-s}^{+s} |d_{m',m}^{(s)}(\theta)|^{2q} \right) . \quad (43)$$

The quantum-mechanical expressions for the conditional q -entropies are then written as

$$H_q(X_j|X_{j-1}) = F_q^{(s)}(\Delta\theta) , \quad H_q(X_n|X_1) = F_q^{(s)}(\theta) . \quad (44)$$

It is useful to compare the characteristic quantity (19) with the entropic scale $\ln_q(2s+1)$. The latter gives the maximum of the q -entropy supported on $2s+1$ points. Following [30], we will consider the relative quantity

$$\mathcal{R}_q := \frac{\mathcal{C}_q}{\ln_q(2s+1)} = \frac{F_q^{(s)}(\theta) - (n-1)F_q^{(s)}(\Delta\theta)}{\ln_q(2s+1)} . \quad (45)$$

Due to (19), the hypotheses of macroscopic realism and noninvasive measurability lead to the conclusion $\mathcal{R}_q \leq 0$. Positive values of (45) will imply that predictions of quantum mechanics are not compatible with these hypotheses. Since our aim is to focus on variations of the parameter q , we consider only the simplest choice of s and n .

On Figure 1, we show violations of the q -entropic Leggett–Garg inequalities for the spin $s = 1/2$ and the number of observations $n = 3$. The curves are related to the values $q = 1.0; 1.2; 1.5; 1.8; 2.4$. The standard choice $q = 1$ considered in [32] is included for comparison. We see that the curve maximum goes to larger values of θ with growth of q . There is some extension of the domain, for which $\mathcal{R}_q > 0$. For $q > 2.4$, however, such an extension becomes negligible. Nevertheless, the curves of figure 1 clearly show an utility of the q -entropic approach. In this regard, q -entropic inequalities of the Leggett–Garg type are similar to the q -entropic Bell inequalities derived in [30]. Measured results of the experiment with fixed θ does violate the inequality $\mathcal{R}_q \leq 0$ for one values of q and does not for other ones, including the standard case $q = 1$. It is a manifestation of the following fact. Entropic Leggett–Garg inequalities give only necessary conditions that probabilistic models are compatible with the macrorealistic picture.

For larger values of s or n , a similar picture is observed. Here, we refrain from presenting corresponding curves. Instead, we describe some significant points. The above mentioned properties of curves for different q remain valid. In particular, there is some domain, in which q -entropic inequalities give advances in comparison with the standard case $q = 1$. On the other hand, with growing s and n we have seen a decrease of this domain. It may be related with the following fact. As reported in [32], both the strength and the range of violations reduce with the increase of spin value. We also recall that the considered situation corresponds to equidistant time intervals. For experiments with unequal time intervals, the q -entropic approach may give additional possibilities for analyzing data of tests of the Leggett–Garg type. Another question is connected with detection inefficiencies.

Using the Shannon entropies, the writers of [28] considered inequalities of the Bell type in the case of detection inefficiencies. For the q -entropic inequalities, this issue was examined in [30]. It turned out that the q -entropic approach can allow to reduce an amount of required detection efficiency. We shall now consider this question for inequalities of the Leggett–Garg type. In the considered example, we have the probability (37), whence

$$H_q(X_k) = \ln_q(2s+1) . \quad (46)$$

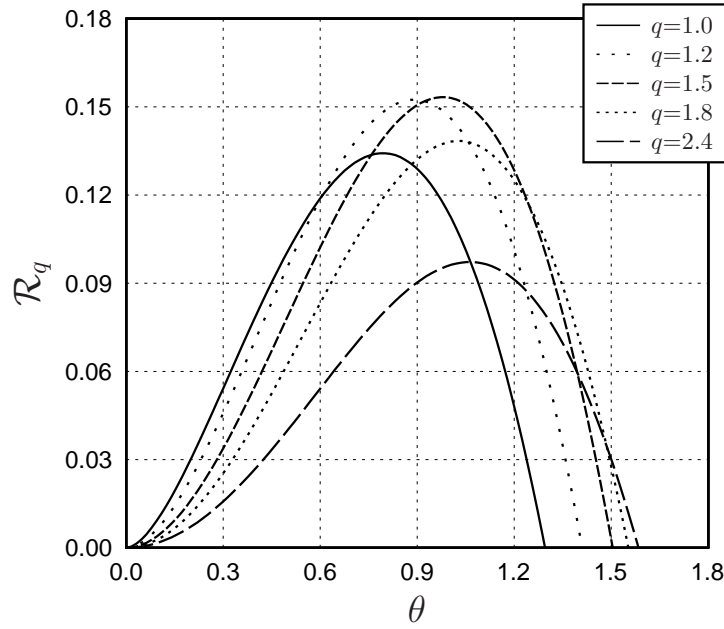


FIG. 1: The relative quantity \mathcal{R}_q versus θ for $s = 1/2$, $n = 3$, and five values of q , namely $q = 1.0; 1.2; 1.5; 1.8; 2.4$. For each value of q , only positive values of \mathcal{R}_q are shown.

TABLE I: The values of the ratio (48) for $\eta = 0.99$ and several q in the case $s = 1/2$, $n = 3$, and $\theta = 0.9$.

q	1.0	1.1	1.2	1.4	1.6	1.8	2.0	2.2	2.4	3.0	5.0	8.0	10.0
$r_q(0.99)$	0.711	0.504	0.386	0.266	0.212	0.186	0.173	0.167	0.165	0.174	0.259	0.587	1.364

Then the additional term (31) reads

$$\Delta_q(\eta) = (n-2) \left\{ \eta^q (\eta^q + 2(1-\eta)^q - 1) \ln_q(2s+1) + (\eta^q + (1-\eta)^q) h_q(\eta) \right\}. \quad (47)$$

The characteristic quantity \mathcal{C}_q is given by the numerator of (45). The q -entropic inequality (19) claims $\mathcal{C}_q \leq 0$. Using measured data, we will actually deal with the quantity (30). As was mentioned above, we must confide that the violating term $\eta^{2q} \mathcal{C}_q$ is sufficiently large in comparison with the additional term (47). To do so, we introduce their ratio

$$r_q(\eta) := \eta^{-2q} \mathcal{C}_q^{-1} |\Delta_q(\eta)|, \quad (48)$$

which is restricted to the case $\mathcal{C}_q > 0$. Let us consider this ratio in our case $s = 1/2$ and $n = 3$. We take $\theta = 0.9$, when the strength of violations is large for several values of q (see Fig. 1). We have calculated $r_q(\eta)$ versus η for such values of q . With respect to η , we especially focus an attention on values, which are close to 1 from below. For fixed q , the ratio $r_q(\eta)$ decreases with such η almost linearly, up to the inefficiency-free value $r_q(1) = 0$. Due to almost linear dependence, we can describe each case by the value of (48) for some η , say, for $\eta = 0.99$. Approximately, we can use $r_q(\eta) \approx 10^2 r_q(0.99) (1 - \eta)$ within a range of linear behavior. In Table I, the value $r_q(0.99)$ is presented for $\theta = 0.9$ and several values of q .

In general, the required amount of efficiency seems to be very high. At the same time, the value $r_q(0.99)$ essentially depends on q . Initially, this value quickly decreases with $q > 1$. It then becomes increasing for sufficiently large q . Among q -entropic inequalities of the Leggett-Garg type, the choice $q = 2$ is convenient. First, both the strength and range of violations are significant. Second, the ratio (48) is small for $\eta > 0.99$ (see Table I). Third, properties of the q -entropies are mathematically simpler in the case $q = 2$. We have already reported such reasons in [30], where q -entropic inequalities of the Bell type were obtained. Studying the CHSH and KCBS scenarios, the q -entropic Bell inequalities were shown to be expedient. Using q -entropic inequalities, we can also reach new possibilities for analyzing data of the Leggett-Garg tests.

V. CONCLUSIONS

We have formulated inequalities of the Leggett–Garg type in terms of the q -entropies. For all $q \geq 1$, such inequalities follow from the existence of some joint probability distribution for outcomes of measurements at different instants of time. It turned out that quantum mechanics predicts violations of an entire family of q -entropic inequalities of the Leggett–Garg type. We illustrated violations with the example of quantum spin systems. The spin- s system has been prepared initially in the completely mixed state. Entropic Leggett–Garg inequalities give only necessary conditions that probabilistic models are compatible with the macrorealism in the broader sense. We showed that the presented inequalities allow to widen a class of situations, in which an incompatibility with the macrorealism can be checked. Both the strength and range of violations can be increased by adopting appropriate values of the entropic parameter q . We also formulated q -entropic inequalities of the Leggett–Garg type in the case of detection inefficiencies. If we use q -entropic inequalities, then the required amount of efficiency can somehow be reduced. We hope that the presented formulation could be useful in analysis of recent experiments to test Leggett–Garg inequalities.

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